Gravity 1

Some Essential Facts

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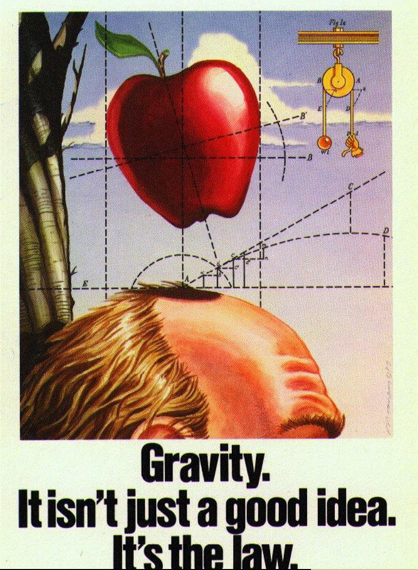
Potential Fields Geophysics: Week 1

Objectives for Week 1



- Review the basics of Newton's Laws
- Learn about Gravity Units
- Understand the major causes of planetary-scale variation in gravity

Newton's Law

Newton's law of gravitation:

$$F = \frac{G m_1 m_2}{r^2}$$

where:

F is magnitude of gravitational force.

m_1 and m_2 are masses.

r is the distance between the two masses.

G is the gravitational constant 6.67×10^{-11} $\text{Nm}^2\text{kg}^{-2}$

Units of F are $\text{N} = \text{kgms}^{-2}$.

Newton's law does not describe why gravity exists or how it works. Instead, his law describes the magnitude of gravity in terms of gravitational force with the famous inverse square law. In its classical form, the law shows force that exists between two masses in terms of their distance apart and the gravitational constant. The masses are "point" masses, so their geometry is ignored and the distance between them is the distance from point to point.

Newton did not know the value of the gravitational constant. This was later measured by Cavendish and refined, but it is still imprecisely known. We think the gravitational constant is the same everywhere in the universe, but cannot be sure since we lack a precise theory of why gravitational force exists. In geophysics, the main thing to remember about the gravitational constant is it's units. The constant works out so that the force is directly proportional to mass and inversely proportional to distance, yielding the constant's non-intuitive units. Most calculations are done in the MKS system, as shown here, and then converted to a different system of units, as described later.

Using Newton's Law

Examples

What is the magnitude of gravitational force between two touching billiard balls?
Given:
Centers are 7.5×10^{-2} m apart
Mass of each ball is 0.225 kg

$$F = \frac{6.67 \times 10^{-11}(0.225)^2}{(7.5 \times 10^{-2})^2} \approx 6 \times 10^{-10}\text{N}$$

What is the gravitational force acting between one billiard ball resting on the Earth's surface and the Earth (mass of Earth, $M_E = 5.9742 \times 10^{24}$ kg; radius of Earth, $R_E = 6.378 \times 10^6$ m)?

$$F = \frac{6.67 \times 10^{-11}(0.225)(5.9742 \times 10^{24})}{(6.378 \times 10^6)^2} \approx 2\text{N}$$

Of course the gravitational force has magnitude and direction. If there are more than two billiard balls, the gravitational force is calculated between each pair and summed. This is a vector addition because the direction and magnitude of the attraction varies. Using the illustration above, consider gravitational force acting on the 10 ball due to the other three balls.

Forces sum

Gravitational (and magnetic) forces sum. This is extremely important because it means the total gravitational force acting on a point is the sum of all gravitational forces acting on that point. The total distribution of masses affect the gravitational force at any given point.

Consider a billiard ball resting on the surface of the Earth at solar noon (the sun is directly overhead). How much is the magnitude of the gravitational force acting on the ball reduced by the presence of the sun (mass Sun, $M_S = 2 \times 10^{30}$ kg, distance to Sun, $D_S = 1.5 \times 10^{11}$ m)?

Gravitational acceleration

Newton's second law of motion is:

$$F = ma$$

where:
m is mass.
a is acceleration.

By combining the universal law of gravitation with Newton's second law of motion, the acceleration of m_2 due to its attraction by m_1 is:

$$\begin{aligned} F &= m_2 a \\ &= \frac{G m_1 m_2}{r^2} \\ a &= \frac{G m_1}{r^2} \end{aligned}$$

Galileo presented his work on motion as a dialogue among three characters, Salviati, Sagredo and Simplicio. A critical statement in the dialogue is by Sagredo, who says in response to a misstatement by Simplicio, the simpleton in Galileo's dialogue representing the Aristotelian point of view: "But I, Simplicio, who have made the test, can assure you that a cannon ball weighing one or two hundred pounds, or even more, will not reach the ground by as much as a span ahead of a musket ball weighing only half a pound, provided both are dropped from a height of 200 cubits." Some people (e.g., Stephen Hawking) have called this statement the birth of modern science – certainly experimental science. Basically, Newton was able to formulate physical laws (shown at left) that quantified Galileo's observations about motion. Although the gravitational force acting between the Earth and the "two hundred pound cannonball" is much greater than the force acting between the Earth and the "half pound musket ball", the gravitational acceleration is the same (in both cases m_2 cancels and the mass m_1 is the mass of the Earth). The quantification is important – from Hawking (God Created the Integers): "Where Galileo had shown that objects are pulled toward the center of the earth, Newton was able to prove that this same force, gravity, affected the orbits of the planets."

Gravitational Acceleration

Considering gravitational acceleration, a, rather than force, F, means we can think about gravity at a point, anywhere in space, without reference to the mass at that point (m_2). That is, gravitational acceleration is defined at any point relative to the masses of the Earth, Sun, Moon, and the mass-distribution of all other objects.

Units of gravitational acceleration

For a bulk (homogeneous, spherical) Earth model, gravity above the surface ($R > R_E$):

$$g = \frac{GM_E}{R^2}$$

where:
g is gravitational acceleration.
G is the gravitational constant.
M_E is the mass of the Earth.
R is the distance from the center of the Earth.
R_E is the radius of the Earth.

The units of acceleration are m s^{-2}. One Gal = 0.01 m s^{-2}. One mGal (the most common unit used to describe variations in gravity at the Earth's surface) is 10^{-5} m s^{-2}. Average acceleration at the surface of the Earth is 9.8 m s^{-2} or 980 000 mGal.

Another commonly used unit, especially to describe changes in gravity with time or very small change in gravity with space, is the microgal. 1 μGal is 10^{-8} m s^{-2}.

Examples

What is the gravitational acceleration at the surface of the Earth due to the Moon when it is directly overhead (mass of moon, $M_M = 7.3 \times 10^{22}$ kg. approximate distance to moon, $R_M = 3.8 \times 10^8$ m)?

$$\begin{aligned} g &= \frac{(6.67 \times 10^{-11})(M_M)}{R_M^2} \\ &= 0.000033720 \text{ms}^{-2} = 3.4 \text{mGal} \end{aligned}$$

What is the vertical gradient in gravity near the surface of the earth?

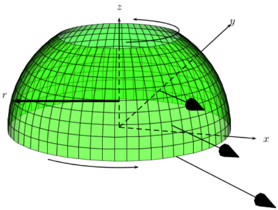
$$\begin{aligned} g &= \frac{GM_E}{R^2} \\ \frac{dg}{dR} &= \frac{-2GM_E}{R^3} = \frac{-2}{R_E} g \end{aligned}$$

since $g = 9.8$ m s^{-2}, and $R_E = 6.378 \times 10^6$ m, $\frac{dg}{dR} = -0.000003073\,\text{s}^{-2}$, or -0.3073 mGal/m. So increasing elevation by one meter near the surface of the Earth decreases gravity by about 0.3 mGal.

Variation in gravity with rotation and shape

As hown in the examples on the previous slide, gravity varies with spatial position and with time, because the distribution of mass (planets) in the solar system changes with time. Also, gravity changes with distance from the center of mass of the Earth, illustrated by the change in $\frac{dg}{dR}$ near the surface. These observations lead us to the point that we have to consider all sources of potential variation in gravity on the surface of the Earth, since we wish to discern gravity changes due to geologic - tectonic - volcanological processes. So far, our calculations have treated the Earth as a static sphere. The fact that the Earth is not static, but rotates, has an important impact on gravity at the surface. Centrifugal force,the same force you feel in a car turning a sharp corner, works against gravitational acceleration and decreases the total acceleration at the surface, especially at low latitudes. Similarly, the Earth is not a sphere, exactly, but an oblate ellipsoid of revolution (fatter at the equatorial waistline). This also affects gravity at the surface because the surface of the Earth is further from the center of mass (R_E is larger) at the equator than at the poles and also because there is more dense mantle and core material between the surface and the center of the Earth at the equator, due to the fatter waistline. Altogether, there are three factors that change gravity at the surface due to the rotation and shape of Earth, or other planets: centrifugal acceleration due to rotation, change in distance from the center of mass, and overall change in mass distribution (more mass at the equator).

Centrifugal acceleration



Centrifugal acceleration varies with distance from the geographic axis of the Earth (r, and so with latitude, ϕ. For a spherical Earth, the change in r with ϕ is:

$$r = R_E \cos\phi$$

The change in gravity with latitude on spherical Earth can be expressed in terms of latitude:

$$g_\phi = g_{pole} - \omega^2 R_E \cos\phi$$

where ω is the angular frequency of rotation of the Earth, $\omega = 2\pi/T$, where T is the period of the earth's rotation (24 hr). This equation shows that for a spherical rotating planet, gravity is maximum at the poles ($r = 0$) and is minimum at the equator.

Examples

What is the centrifugal acceleration at the Earth's equator?
Without going through the derivation:

$$a = \frac{v^2}{r}$$
$$v = \frac{2\pi^2 r}{T}$$
$$a = \frac{4\pi^2 r}{T^2}$$

where:
a = centrifugal acceleration, r is distance from the geographic axis of rotation (the Earth's radius at the equator), and T is the rotation period (24 hr). For the Earth at the equator, centrifugal acceleration is about $a = 0.03$ m s^{-2}, or about 3 Gal.

Change in radius with latitude

Now consider the change in gravity due to the shape of the Earth. This involves calculating the change in gravity with change in Earth radius as a function of latitude. The oblate ellipsoid can be thought of as flattened:

$$f = \frac{R_{equator} - R_{pole}}{R_{equator}}.$$

The magnitude of flattening, f, is now very well determined from observing artificial satellite orbits around the Earth, and is fixed for the WGS84 ellipsoid as $f = 1/298.257223563$. Newton first discussed the flattening of the Earth, and estimated a value of $1/230$. The radius of the Earth at a given latitude is given by;

$$R_E = R_{equator}(1 - f \sin^2 \phi).$$

Again for the WGS84 ellipsoid, $R_{equator} = 6378137.0\,\mathrm{m}$.

Examples

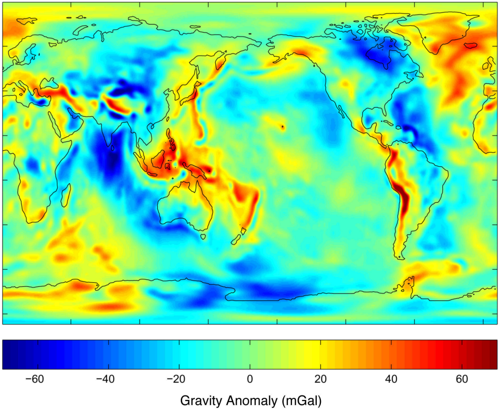
Using the WGS84 ellipsoid, the estimated radius of the Earth at the equator is $R_{equator} = 6378137.0\,\mathrm{m}$ and at the pole is $R_{pole} = 6356752.31\,\mathrm{m}$. What is the difference in gravity at the surface of spherical planets with Earth's mass and these two different radii?

$$\begin{aligned} g_E &= \frac{GM_E}{R_E^2} \\ g_{equator} &= 9.794319911\mathrm{m\ s}^{-2} \\ g_{pole} &= 9.861319102\mathrm{m\ s}^{-2} \end{aligned}$$

This is about 7 Gal difference. Note that the actual effect is less because the exact calculation is for gravity on an oblate ellipsoid, accounting for the redistribution of mass as well as radius. That is, the extra Earth mass at the equator compared to the poles decreases the overall difference in gravity. In reality, the affect of rotation is larger than the affect of shape.

large-scale variations in Gravity on Earth

To summarize, the Earth's gravity field is primarily described by the gravitational attraction of a sphere, with significant departures due to rotation, change in radius with latitude, and to a much lesser extent due to tides (relative position of the sun and moon). Changes in gravity at the Earth's surface are also related to changes in elevation and related factors. Even when these factors are accounted for, variation in the gravity field persists, although these differences are measured in the mGal range, rather than in the Gal range.



Note that this gravity map was collected above the Earth's surface during the Gravity Recovery and Climate Experiment (GRACE) mission. It is the most accurate map yet of Earth's gravity field.

Gravity varies globally

The GRACE map shows that gravity varies across the globe, once the huge effects of rotation and Earth shape are accounted for. These anomalies are related to large-scale differences in mass distribution within the Earth.

Summary

Newton's laws of motion describe the planetary-scale features of the Earth's gravity field.

- Newton's law of universal gravitation explains the gravitational force, and acceleration, that exists between any two masses, including the Earth and any object in the "Earth's gravity field". The gravity field due to the Earth, or other planets, can be described by the change in acceleration from place to place. If only the magnitude of acceleration is shown, then the field is described as a scalar field. If direction is shown, the field is described as a vector field. We define direction (vertical) in terms of the orientation of the vector field of gravitational acceleration.
- The units of gravitational acceleration are usually mGal. $10^5\,\text{mGal} = 1\,\text{ms}^{-2}$. Note that a common error in calculating gravity is to forget to covert MKS to mGal!
- Newton's laws successfully describe large scale variations in gravity on the surface of Earth due to (a) change in latitude (radius, rotation), (b) change in elevation, (c) tidal effects (position of Sun and Moon).
- Gravity anomalies persist at a huge range of scales in the Earth, due to variations in the distribution of mass.

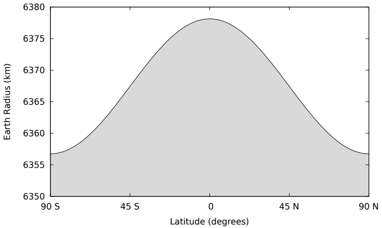
Additional Resources: Read Blakely, Chapter 1, 3, 7

End of Module Questions

1. Gravity changes at the surface of the Earth due to the position and mass of the sun and the moon. Consider a point on Earth's surface that has the moon directly overhead (lunar noon). In this geometry, the Moon causes an upward acceleration, effectively decreasing the gravitational acceleration at that point due to the Earth's mass alone. Roughly 12 hr later, the Moon is exactly on the opposite side of the Earth (lunar midnight) effectively increasing the gravitational acceleration at the same point on the Earth's surface due to the Earth's mass alone. Calculate the change in gravitational acceleration at the point (in mGal) between the lunar noon and lunar midnight. Assume a spherical Earth of radius $R_E = 6.378 \times 10^6$ m, the distance from the center of the Earth to the Moon is $R_M = 3.8 \times 10^8$ m and the lunar mass is 7.3×10^{22} kg.
2. Paleontological data and astrophysical models suggest there were about 423 days in one Earth-year, 600 million years ago. No data suggests the Earth's orbit has changed, so the change is attributed to tidal friction slowing the Earth's rotation with time. What is the increase in gravity at the equator from 600 Ma to the present due to this change in rotation, assuming no change in the Earth's shape? By extrapolation, the period of the Earth's rotation around 4 billion years ago was 6 hr. What is the difference in gravity at the equator due to this change in rotation? Express your answer in mGal.
3. Plot a graph showing three curves: of the change in gravity as a function of latitude due to change in angular velocity, the change in gravity due to the flattening of the Earth, and the sum of these two graphs. Discuss the plot, which factor has the largest affect on gravity? You can use any method you wish to make this plot. A simple way to get started is using gnuplot (described in the supplementary material).

Plotting with gnuplot

A simple way to plot functions is to use gnuplot (open-source, freely available software, mostly used by scientists using linux systems). Here is an example gnuplot code to plot a function described in this module (radius of the Earth as a function of latitude).



The entire code that generates this plot is on the next page

Plotting with gnuplot


```
reset
set termoption dash

#equatorial radius (m), WGS84
Re = 6378137.0

#flattening
f = 1/298.25722356

#radius in km
r(x) = Re * (1-f*sin(x)*sin(x)) / 1000.0

# Axes label
set xlabel 'Latitude '
set ylabel 'Earth Radius (km)'
# Axes ranges
set xrange [-pi/2:pi/2]
set yrange [6350:6380]
# Axes tics
set xtics ('90 S' -pi/2,  '45 S' -pi/4, 0, '45 N' pi/4, '90 N' pi/2 )
set ytics 5
set tics scale 0.75

set style fill transparent solid 0.15
set key Left
plot       r(x) with filledcurve y1=0  lt rgb "gray0" notitle

set term pdf enhanced dashed
set output "Earth_radius.pdf"
replot
```