

Gravity 3

Gravitational Potential and the Geoid

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Potential Fields Geophysics: Week 2

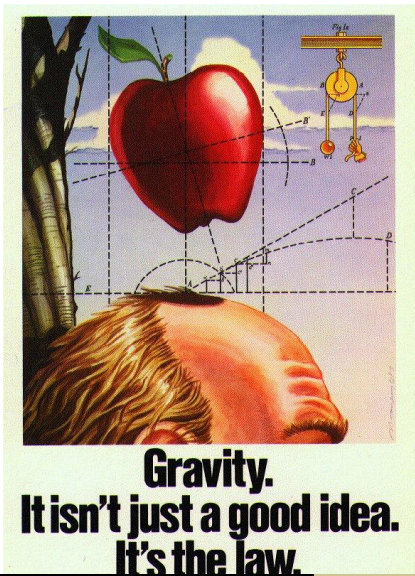
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Objectives

Vectors

Gravitational Potential

Geoid



- Gravity as a vector
- Gravitational Potential
- The Geoid

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We can write Newton's law for gravity in a vector form to account for the magnitude and direction of the gravity field:

$$\vec{g} = -\frac{Gm_E}{r^2} \vec{r}$$

where:

\vec{g} is the gravitational acceleration

m_E is the Earth's mass.

r is distance from the center of mass (e.g., the Earth).

\vec{r} is a unit vector pointed away from the center of mass

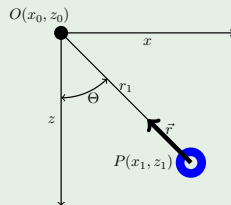
G is the gravitational constant ($\text{m}^3 \text{kg}^{-1} \text{s}^{-2}$)

In this vector form we can think of gravitational acceleration in directions other than toward or away from the mass. Note that \vec{r} is defined as pointing away from the center of mass and in the direction of increasing r , hence the minus sign which could be ignored when we were only concerned with the magnitude of g :

$$g = |\vec{g}| = \frac{Gm_E}{r^2}$$

At right the "z" component of gravity is calculated for point O due to a mass at point P .

Examples



$$g_r = -\frac{Gm_P}{r_1^2} \vec{r}$$

$$\vec{r} = \frac{z_0 - z_1}{r_1}$$

$$g_z = -\frac{Gm_P}{r_1^2} \frac{z_0 - z_1}{r_1}$$

$$g_z = \frac{Gm_P(z_1 - z_0)}{r_1^3}$$

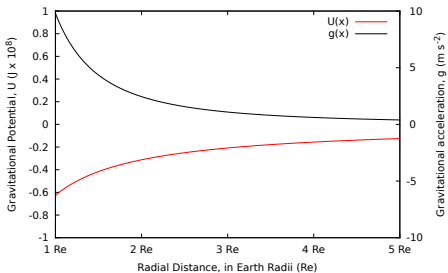
Gravitational potential

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The gravitational potential, U , is a scalar field

$$\begin{aligned}
 U &= \int_{\infty}^R \vec{g} \cdot d\vec{r} \\
 &= \int_{\infty}^R \frac{GM}{r^2} dr \\
 &= -\frac{GM}{R}
 \end{aligned}$$

The signs are tricky. Note $\vec{r} \cdot d\vec{r} = -dr$, that is, \vec{r} and $d\vec{r}$ are of opposite sign (point in opposite directions). Prove to yourself that the MKS unit of gravitational potential is Joules/kg. Gravitational potential is the potential energy per unit mass.



Unlike gravitational acceleration, gravitational potential decreases closer to the surface of the Earth. This means negative work is done by an object falling toward the surface. Positive work is done moving an object away from the surface (it is easier to fall off a cliff than to climb up one!). Both U and g converge to zero at large distances.

Gravity is a potential field

The integral relationship between the vector of gravitational acceleration and the scalar gravitational potential makes gravity a “potential field”. Gravitational acceleration is the gradient of the potential:

$$\vec{g} = -\frac{GM}{r^2} \vec{r} = \frac{\partial}{\partial \vec{r}} \frac{GM}{r} = -\frac{\partial}{\partial \vec{r}} U = -\text{grad } U = -\nabla U$$

The gradient of the potential

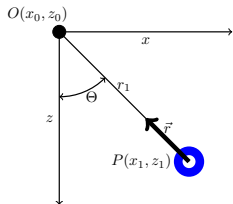
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Let's return to the example using the point mass m_P at point P . The scalar gravitational potential at point O is the work required to move an object from infinity to O . The potential is related to gravitational acceleration by the integral shown on the previous slide. The next question is: how does U vary across the area around P , in this case in the $x - z$ plane? We can think of this as the change of U in the x or z directions, that is, $\partial U / \partial x$ and $\partial U / \partial z$.

Use the chain rule to relate the partial derivative in the r direction to the derivative in the x and z directions:

$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial r_1} \frac{\partial r_1}{\partial z} = -\frac{Gm_P}{r_1^2} \times \frac{\partial r_1}{\partial z}$$

$$\frac{\partial r_1}{\partial z} = \frac{1}{2} \left[(z_1 - z_0)^2 + (x_1 - x_0)^2 \right]^{-\frac{1}{2}} \times 2(z_1 - z_0)$$

$$\frac{\partial U}{\partial z} = -\frac{GM}{r_1^2} \frac{z_1 - z_0}{r_1} = \frac{Gm_P(z_0 - z_1)}{r_1^3}$$

Comparing this to the answer from two slides back:

$$\frac{\partial U}{\partial z} = -g_z$$

$$\frac{\partial U}{\partial x} = -g_x$$

Gravity can vary on an equipotential surface

A surface along which U is constant is an equipotential surface. No work is done against gravity moving on an equipotential surface, but gravity can vary along an equipotential surface because $\frac{\partial U}{\partial z}$, where z is defined as vertical, need not be constant.

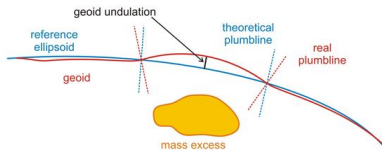
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We have already seen that for a “point” mass, or outside a homogeneous sphere, the potential varies with radial distance only:

$$U = -\frac{GM}{R}.$$

So, $\frac{\partial U}{\partial z} = \text{constant}$ on such an equipotential surface (gravity is constant at a given value of R). The actual Earth is not homogeneous. Earth has mass anomalies, U is not constant at a given R , and $\frac{\partial U}{\partial z} \neq \text{constant}$, so gravity varies along an equipotential surface for the Earth.

The variation in an equipotential surface for the Earth can be thought of in terms of variation of its height. Since potential energy, $U = gh$ on an equipotential surface, and U is constant by definition, any change in gravity corresponds to a change in height, h .

This change in height of the equipotential surface has to be referenced to something. For Earth, the reference ellipsoid is the best-fit ellipsoid to the figure of the Earth at mean sea-level. The geoid is the equipotential surface that varies around this reference ellipsoid. The height of the geoid is the difference in height, or geoid undulation, from the reference ellipsoid at any given location. The height of the geoid, and the value of gravity on the geoid, varies because the distribution of mass inside the Earth is not uniform. Vertical is defined as normal to the geoid, so the orientation of vertical also varies with respect to the reference ellipsoid.

At sea, the surface of the ocean corresponds to the geoid. Changes in mass distribution within the Earth cause changes in the height of the geoid, so there are literally changes in the height of the sea from place-to-place, with respect to the reference ellipsoid. Most satellites orbit on an equipotential surface, so their height (say distance from the surface of the reference ellipsoid) also undulates on an equipotential surface mimicking the shape of the geoid.

The geoid

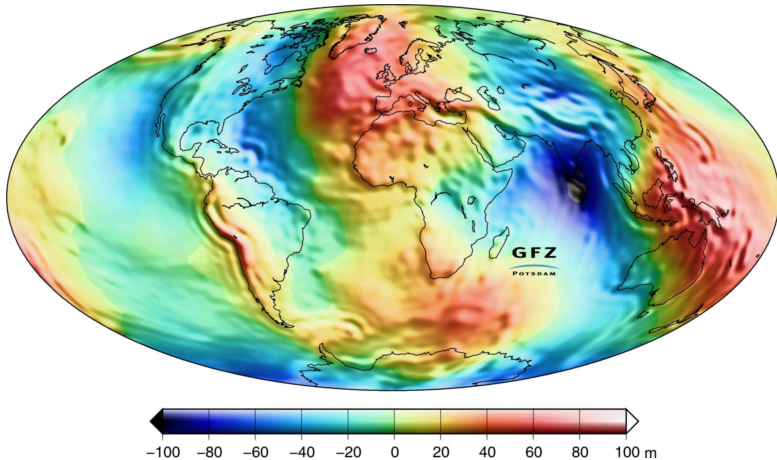
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The Earth's geoid as mapped from GRACE data and shown in terms of height above or below the reference ellipsoid. As time goes on, the geoid has been mapped with greater and greater definition.

What sort of mass distribution causes a change in the Earth's geoid?

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Examples

Consider a geoid anomaly of +50 m on the order of 2000 km in width. What sort of excess mass might cause this geoid anomaly? Let's simplify the problem by considering the excess mass to be in the mantle and of spherical shape. To raise the geoid $h = 50$ m, the gravitational potential on an "undisturbed Earth" at the surface must equal the potential at +50 m once the excess mass is added:

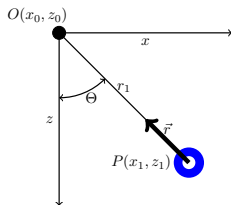
$$-\frac{GM_E}{R_E} = -\frac{GM_E}{R_E + h} - \frac{GM_{excess}}{r_{excess} + h}$$

where M_E is the mass of the Earth, R_E is the radius of the Earth, M_{excess} is the excess mass associated with the geoid anomaly, r_{excess} is the depth to the center of the excess mass, and h is the height of the geoid anomaly. There is one equation and two unknowns (the excess mass and the depth to the center of the excess mass). If we assume the depth to the center of excess mass is 1000 km, prove to yourself that the excess mass creating the geoid height anomaly is about $M_{excess} = 7 \times 10^{18}$ kg. If the excess mass is spherical, then:

$$M_{excess} = \frac{4}{3}\pi a^3 \rho_{excess}$$

where a is the radius of the spherical excess mass and ρ_{excess} is its excess density (or density contrast with the surrounding mantle). If $a = 5 \times 10^5$ m then $\rho_{excess} = 14 \text{ kg m}^{-3}$. The mantle density on average at 1000 km depth is on order of 4000 kg m^{-3} . Geoid anomalies of ten's of meters height and thousands of kilometers width seem to be related to very small perturbations in this density, possibly associated with changes in water content of the mantle, or other geochemical differences, and temperature.

Answer the following questions using the diagram at right. As before, a point mass, m_P is located at P and we are concerned with the gravity and gravitational potential at point O due to the mass at point P .



- 1 Rewrite the equation for gravitational acceleration in the z direction (g_z) due to the mass at point P , only in terms of the constants G and m_P and the variables x and z (that is, eliminate the variable r_1 from the equation).
- 2 Using the equation you derived in question 1, graph the change in g_z with x along a profile across the point P . Assume values for the mass at the point and its depth.
- 3 Now consider the same problem in three dimensions, that is $r_1^2 = x^2 + y^2 + z^2$. Assume that $U = 1/r_1$, since $-GM$ is constant. Show that:

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

This is Laplace's equation. It means that outside the mass, the gravity field is conserved, so the field varies in a systematic way. This fact is highly useful for calculating expected anomalies and for filtering gravity maps.