Gravity 4

Gauss, Excess Mass, Simple Shapes

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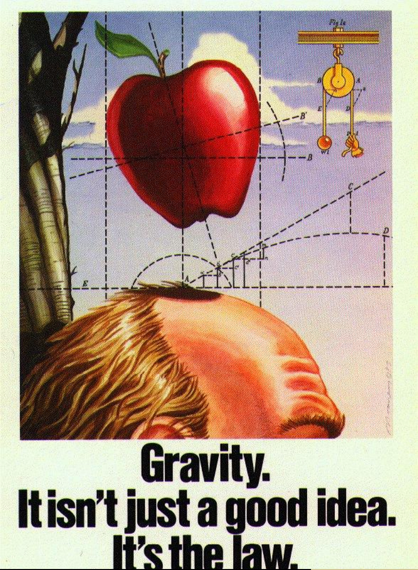
Potential Fields Geophysics: Week 4

Objectives for Week 4



- Learn about Gauss's law
- Excess Mass
- Divergence
- Gravity due to simple shapes

Gauss's law

An equipotential surface, like the geoid, undulates in shape because of mass heterogeneities. The gravitational potential can be reformulated to reflect the distribution of many masses:

$$U = \sum_{i=1}^{N} U_i = \sum_{i=1}^{N} -\frac{Gm_i}{r_i}$$

where:
N is the number of individual masses,
m_i is the i^{th} mass, located at distance r_i,
G is the gravitational constant.

In practice, individual masses inside the Earth are volumes, V, of similar density, ρ. It is the density contrast between masses that defines them. In the limit:

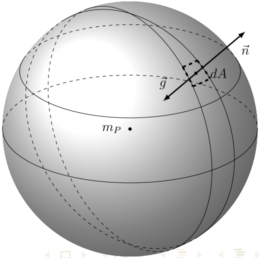
$$U = -G \int_V \frac{\rho dV}{r}$$

This integral is a key part of Gauss's law, which relates the total mass of an anomalous body to the integrated gravity field produced by the body. The mass distribution is described by the above equation. How is the integrated gravity field calculated?

Consider any mass distribution (like the Earth); this mass distribution can be completely contained by a surface (like an equipotential surface). The sum of the gravity field across the entire surface is:

$$\oint_S \vec{g} \cdot \vec{n} dA$$

The contour integral symbol is shorthand indicating that the integral includes the entire surface area, which is divided into small areas, dA. The unit vector, $\vec{n}$, is normal to the surface and directed *outward* (away from the mass) at each dA.



Gauss's law

A physical analogy might help. Consider a well pumping groundwater out of a thick saturated sandstone. Imagine a hypothetical surface that completely encloses the well; in a steady-state, all groundwater pumped from the well must pass through this surface. The total water pumped from the well in any given time (say one second) is equal to the amount of water passing through the hypothetical enclosing surface in the same amount of time. The "field" in this case is the velocity of water pulled toward the well across the surface, the potential is the pressure gradient that drives the water at this velocity, and the contour integral yields the total flux of water across the surface.

Suppose the sandstone aquifer is so uniform that a spherical surface exists, all across which the groundwater velocity is $2\,\mathrm{m\,s^{-1}}$ toward the well. The surface area of this hypothetical sphere is $4\pi r^2$, so the well must be pumping water at a rate of $8\pi r^2\,\mathrm{m^3\,s^{-1}}$. The velocity of water integrated across the entire enclosing spherical surface is equal to the flux of water from the well, regardless of the value of r.

For a point mass with the spherical enclosing surface located at r, the solution to the integral is just the surface area of the sphere multiplied by the gravity at r:

$$\oint_S \vec{g}\cdot\vec{n}\,dA = -4\pi r^2\frac{GM}{r^2} = -4\pi G\int_V \rho\,dV.$$

In other words, the integral of the gravity across the entire enclosing surface is proportional to the mass enclosed by the surface, regardless of the value of r. This is Gauss's law. It turns out to be applicable regardless of the mass distribution within the enclosing surface, or the shape of the enclosing surface. Because we know G very well, the total mass can be determined by measuring the gravity field, without knowing anything about the mass distribution. From satellites (e.g., GRACE) we can measure gravity everywhere on a surface and so determine the mass of the Earth, despite the fact that mass is heterogeneously distributed in Earth. In fact, after Cavendish estimated G, a big activity of geophysicists was measuring g around the globe to improve estimates of the Earth's mass.

Excess mass

Gauss's law:

$$\oint_S \vec{g} \cdot \vec{n} dA = -4\pi GM$$

allows us to calculate the total excess mass that produces a gravity anomaly, regardless of the distribution of the mass. Green found this solution to the excess mass within the Earth using a hemisphere for the hypothetical surface, S. Assume the flat top of the hemisphere is the surface of the Earth ($z = 0$) The surface integral can be divided into the flat top and the hemisphere. By assuming the radius of the hemisphere is extremely large, some tricks are played with the limits of integration to simplify the problem. The equation for the surface integral becomes:

$$\oint_S \vec{g} \cdot \vec{n} dA = \int_{z=0} \int g(x,y) dx dy + \int_{\phi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} g(r,\theta) sin\theta d\theta d\phi = 4\pi GM.$$

This reduces to

$$\int_{z=0} \int g(x,y) dx dy = 2\pi GM,$$

So the anomalous mass is:

$$M = \frac{1}{2\pi G} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) dx dy$$

In practice, the anomalous mass is found by numerically integrating the gridded gravity data across an area:

$$M = \frac{1}{2\pi G} \sum_{i=1}^{N} \sum_{j=1}^{M} \Delta g(x,y) \Delta x \Delta y$$

where $\Delta g(x,y)$ is the gravity anomaly, N an M are the number of grid points in the X and Y directions, respectively, and Δx and Δy is the grid spacing in the X ad Y directions.

Example excess mass calculation

Example

The excess mass on a gridded gravity map is:

$$M = \frac{1}{2\pi G} \sum_{i=1}^{N} \sum_{j=1}^{M} \Delta g(x,y) \Delta x \Delta y$$

where $\Delta g(x,y)$ is the gravity anomaly, N an M are the number of grid points in the X ad Y directions, respectively, and Δx and Δy is the grid spacing in the X ad Y directions. For the grid shown, the "background" gravity value is 0.2 mGal, so $\Delta g(x,y)$ is the difference from 0.2 mGal. To calculate the excess mass for gravity data:

1. interpolate the gravity data on to a regular grid
2. find the "background" gravity value
3. for each grid point, subtract the background value and sum
4. multiply the sum by $\frac{1}{2\pi G} \Delta x \Delta y$

$\Delta x = 1\,\text{km}$, $\Delta y = 1\,\text{km}$

•0.20	•0.20	•0.19	•0.20	•0.20
•0.20	•0.12	•0.15	•0.16	•0.20
•0.20	•0.11	•0.05	•0.10	•0.20
•0.20	•0.16	•0.14	•0.17	•0.20
•0.20	•0.19	•0.20	•0.20	•0.20

Prove to yourself that the excess mass for the above grid of gravity data is approximately $M = -1.57 \times 10^{10}$ kg, with the values posted next to each grid point given in mGals. The negative mass indicates that mass is "missing" and the gravity low is caused by the occurrence of lower density material. If the anomaly were caused by an air-filled cave system in limestone ($\rho = 2300\,\text{kg}\,\text{m}^{-3}$), find the volume of the cave system. (Ans $= 6.8 \times 10^{6}\,\text{m}^3$).

Gravity inside a homogeneous planet

What is the gravity field inside a planet? Referring to the figure at right, where R_E is the radius of the planet, what is the gravity at r ($r < R_E$)? Assume the planet is of homogeneous density, ρ. The the total mass of the planet inside r is:

$$M_E = \frac{4}{3}\pi\rho R_E^3,\ m = \frac{4}{3}\pi\rho r^3$$

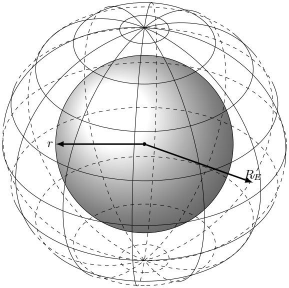
so

$$m = M\frac{r^3}{R_E^3}$$

Given Gauss's law:

$$\begin{aligned}
\oint_S \vec{g}\cdot\vec{n}dA &= -4\pi GM, \\
-4\pi r^2 g &= 4\pi GM\frac{r^3}{R_E^3} \\
g &= G\frac{M}{R_E^3}r
\end{aligned}$$

so gravity varies linearly with r, and $g \to 0$ as $r \to 0$.



The same approach works for finding gravity outside a planet ($r > R_E$).

$$\begin{aligned}
-4\pi r^2 g &= 4\pi GM \\
g &= G\frac{M}{r^2},
\end{aligned}$$

Outside the planet, gravity varies with $1/r^2$, and $g \to 0$ as $r \to \infty$.

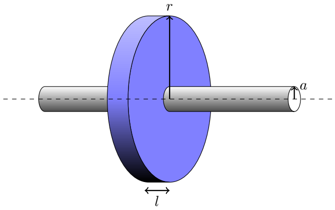
Gravity anomaly due to an infinite horizontal cylinder

Estimate the gravity anomaly due to an infinite horizontal cylinder using Gauss's law:

$$\begin{aligned} \oint_S \vec{g} \cdot \vec{n} dA &= -4\pi GM \\ M_l &= \pi a^2 \Delta\rho \\ -2\pi r l g &= -4\pi G M_l l \\ g = \frac{2GM_l}{r} &= \frac{2G\pi a^2 \Delta\rho}{r} \end{aligned}$$

where M_l is the mass per unit length of the cylinder and $\Delta\rho$ is the density contrast between the cylinder and the surrounding rock. The trick in this case is to realize that the "Gaussian" surface enclosing the cylinder can be expressed in terms of surface area per unit length along the cylinder, as can the mass per unit length. This works if the cylinder is infinite, so the ends are ignored.

The blue disk in this figure represents the hypothetical Gaussian surface, of radius r. Because the vector of gravity due to the cylinder is oriented toward the axis of the cylinder in the plane of the disk, the surface integral is circumference of the blue disk times its unit thickness, l.



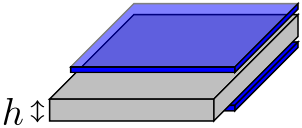
A surprising amount of geology can be investigated with the horizontal cylinder model, not just tunnels and lava tubes. Density contrasts associated with folds (anticline and syncline pairs) are sometimes modeled by horizontal cylinders. Long linear basins are modeled this way as a first approximation, with the surface of the basin corresponding to the top of the cylinder.

Gravity anomaly due to an infinite horizontal slab

What is the gravity anomaly due to an infinite horizontal slab, a horizontal unit that is of such great lateral extent it can be considered to be infinite?

$$\begin{aligned} \oint_S \vec{g} \cdot \vec{n} dA &= -4\pi G M \\ M_S &= S\Delta\rho h \\ -2Sg &= -4\pi G M_S \\ g &= 2\pi G \Delta\rho h \end{aligned}$$

where S is the unit surface area of the slab, and M_S is the mass of the slab per unit surface area. The "Gaussian" surface enclosing the slab is just two times the surface area. One part of the hypothetical "Gaussian" surface is shown as the blue plate above the slab, the other part is the blue plate below the slab. Because the slab is infinite, the edges of the enclosing surface are not considered; only the surface above and below the slab need be considered.



The gravity anomaly due to the infinite horizontal slab does not depend on the distance, r from the slab. We can think of this in terms of the divergence of the gravity field, which we can calculate in terms r:

$$\begin{aligned} \nabla \cdot \vec{g} &= \frac{\partial \vec{g}}{\partial r}\hat{\mathbf{r}} \\ &= \frac{\partial}{\partial r}(2\pi G \Delta\rho h)\hat{\mathbf{r}} \\ &= 0 \end{aligned}$$

Physically, this means that the flux of gravity through the surface does not change with distance. So the gravity anomaly due to the infinite slab is the same 10 m from the slab as it is 1000 m away.

Calculating Divergence

We can see from the infinite slab example that divergence is an important topic used to understand the shape of a vector field, like the gravity field. Consider the vector field:

$$\vec{v} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}.$$

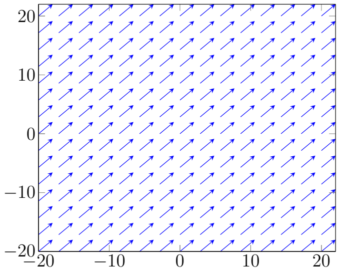
The divergence is the sum of the first derivative in the x direction and the first derivative in the y direction. The divergence of $\vec{v}$ is

$$\begin{aligned} \nabla \cdot \vec{v} &= 2\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} \\ &= 0 + 0 \end{aligned}$$

The divergence is 0 because the first derivative of the vector field is zero in both the x or z directions. This is exactly the same as the gravity field for an infinite slab:

$$\begin{aligned} \nabla \cdot \vec{g} &= 0\frac{\partial}{\partial x} + 2\pi G \Delta\rho h \frac{\partial}{\partial z} \\ &= 0 + 0 \end{aligned}$$

$\vec{g} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, $\nabla \cdot \vec{g} = 0$



The gravity field does not diverge with distance from the slab because the magnitude of gravity does not depend on the distance (say its depth, z). The concept of an infinite slab is actually useful in making elevation corrections to gravity data, so such infinite plates are sometimes called Bouguer slabs, after Pierre Bouguer.

Divergence of flowing magma

Magma rising in vertical volcano conduit is a good example of positive divergence in one dimension. The magma is rising because of the pressure gradient along the conduit. As that magma rises, bubbles nucleate and grow. These bubbles take up extra volume, so the magma must accelerate toward the surface. Suppose the velocity of the magma as a function of distance along the conduit is:

$$\vec{v} = e^x$$

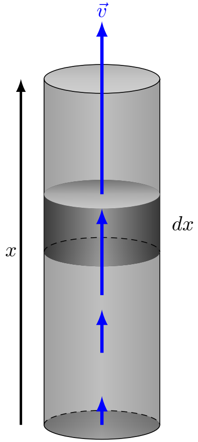
where x is the distance along the conduit. Then:

$$\nabla \cdot \vec{v} = e^x \frac{\partial}{\partial x} = e^x$$

Think of a small slice of the cross sectional area of the conduit, dx. The velocity of the magma is higher leaving dx than entering dx, because of bubble nucleation and expansion within dx. Positive divergence in this example means that the density of the mixture is less with height. The further along the conduit, the larger x, the larger the divergence, meaning this expansion process accelerates toward the surface.

In contrast, if the magma is volatile free and only the pressure gradient drives the magma upward, then $\vec{v}$ is constant along the conduit and:

$$\nabla \cdot \vec{v} = 0$$



A negatively divergent field

Consider a more complicated vector field:

$$\vec{v} = \frac{1}{x^2+y^2}\hat{\mathbf{i}} + \left[\frac{1}{x^2+y^2} - \frac{1}{5y}\right]\hat{\mathbf{j}}$$

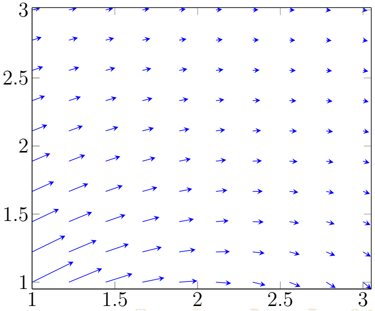
Then:

$$\nabla \cdot \vec{v} = \frac{1}{x^2+y^2}\frac{\partial}{\partial x} + \left[\frac{1}{x^2+y^2} - \frac{1}{5y}\right]\frac{\partial}{\partial y}$$

Prove to yourself that the divergence is:

$$\nabla \cdot \vec{v} = \frac{1}{5y^2} - \frac{2(x+y)}{(x^2+y^2)^2}$$

If you evaluate the divergence at point $(1, 1)$, you will find $\nabla \cdot \vec{v} = -4/5$; the divergence is much smaller, but still negative at point $(3, 3)$. You can see from the plot of this vector field that the vectors get smaller in the direction of flow. If this field describes velocity of air, the air gets more dense in the direction of flow. This pattern is characteristic of a shock wave (e.g., an explosion). Shock waves are negatively divergent. The effect (e.g., air getting dense) is much more pronounced at point $(1, 1)$ than at point $(3, 3)$.



Divergence of a gravity field

Example

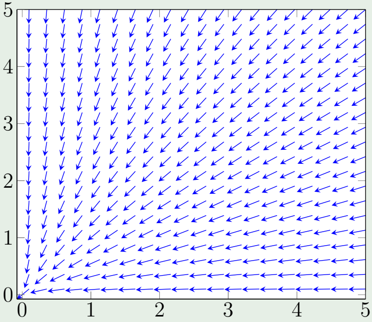
Recall from Module 3 that the vertical component of the gravity field associated with a point mass, m_P is

$$g_z = \frac{G m_P z}{\sqrt{x^2 + z^2}}$$

Then you can show that:

$$\begin{aligned} \nabla \cdot \vec{g} &= \frac{G m_P x}{\sqrt{x^2+z^2}} \frac{\partial}{\partial x} + \frac{G m_P z}{\sqrt{x^2+z^2}} \frac{\partial}{\partial z} \\ &= \frac{G m_P}{\sqrt{x^2+z^2}} \end{aligned}$$

For a point source located at $(0, 0)$, the divergence is larger closer to the origin and decreases with distance, but is always positive (the gravity field is divergent). One way to compare gravity anomalies is to consider their divergence. A broad gravity anomaly will be less strongly divergence that a less broad gravity anomaly.



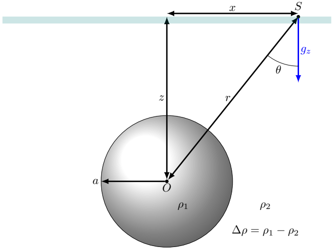
Gravity anomaly due to a buried sphere

It is possible to calculate the exact gravity anomaly due to a wide variety of simple shapes. The simplified geometry of these shapes (spheres, rods, plates) simplifies the analytical solution to find their gravity anomalies (e.g., application of Gauss's law). A surprisingly large number of natural gravity anomalies can be assessed by comparing them to the gravity anomalies of simple shapes. A general form from the vertical component of the gravity anomaly due to a buried mass is:

$$g_z = G \int \frac{dm}{r^2} \cos\theta$$

Note that the vertical component is defined by the Earth's gravity field. It is assumed that the deflection of the equipotential surface (and deflection of the vertical) can be neglected. That is, the anomalous mass is very small compared to the magnitude of the Earth's field. For a sphere with the geometry shown:

$$\begin{aligned} g_z &= g_r \cos\theta \\ &= \frac{GMz}{r^3} \\ &= \frac{4\pi G \Delta\rho a^3}{3} \frac{z}{(x^2+z^2)^{3/2}} \end{aligned}$$



This figure shows the gravity anomaly at point S due to a sphere located entirely below the surface at point O ($z > a$). The Earth's surface is indicated by the thick pale blue line.

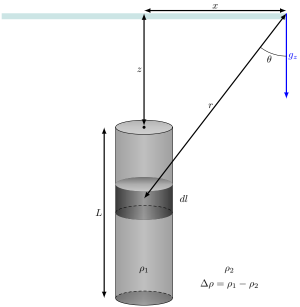
Gravity anomaly due to a thin vertical rod

Find the gravity anomaly due to a thin vertical rod. The rod is "thin" so that it can be assumed the cross sectional area of the rod is not important (gravity due to one edge of the rod is not considered separately from gravity on the other edge).

$$g_z = GA\Delta\rho \int_0^L \frac{dl}{r^2} \cos\theta$$

where A is the cross sectional area of the rod.

$$\begin{aligned}
g_z &= GA\Delta\rho \int_0^L \frac{(z+l)dl}{\left[x^2+(z+l)^2\right]^{1/2}\left(x^2+(z+l)^2\right)} \\
u &= (z+l)^2 \\
du &= 2(z+l)dl \\
g_z &= GA\Delta\rho \int \frac{du}{\left[x^2+u\right]^{3/2}} \\
&= -GA\Delta\rho \left[x^2+(z+l)^2\right]^{-1/2} |_0^L \\
&= GA\Delta\rho \left[\frac{1}{\sqrt{x^2+z^2}} - \frac{1}{\sqrt{x^2+(z+L)^2}}\right]
\end{aligned}$$



Prove to yourself that as $L \to \infty$, the gravity anomaly becomes

$$g_z = \frac{GA\Delta\rho}{\left[\sqrt{x^2+z^2}\right]}$$

Gravity anomaly due to a finite horizontal unit

The gravity anomaly due to a horizontal unit of finite extent in the x direction is:

$$g_z = 2G\Delta\rho h\left(\pi + \tan^{-1}\left[\frac{x}{z}\right] + tan^{-1}\left[\frac{l-x}{z}\right]\right)$$

where:
z is the depth to the top of the horizontal unit

x is the horizontal offset from the edge of the unit

h is the thickness of the offset horizontal unit

l is the length of the horizontal unit

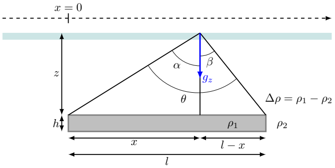
$\Delta\rho$ is the density contrast between the unit and surrounding rock

G is the gravitational constant.

This expression can also be written as:

$$g_z = 2G\Delta\rho h\left[\alpha + \beta\right] = 2G\Delta\rho h\left[\theta\right]$$

Note that the horizontal unit is assumed to extend infinitely in the y direction (in and out of the $x - z$ plane).



Prove to yourself that as $l \to \infty$:

$$g_z = 2G\Delta\rho h\left(\frac{\pi}{2} + \tan^{-1}\left[\frac{x}{z}\right]\right)$$

This simple shape is sometimes referred to as a semi-infinite slab.

Gravity anomaly due to a vertical fault

The gravity anomaly due to an offset of a horizontal unit across a vertical fault is:

$$g_z = 2G\Delta\rho h \left(\pi + \tan^{-1}\left[\frac{x}{z_1}\right] - tan^{-1}\left[\frac{x}{z_2}\right]\right)$$

where:
z_1 and z_2 are depths to the top of the faulted horizontal unit

x is the horizontal offset from the intersection of the fault plane with the surface

h is the thickness of the offset horizontal unit

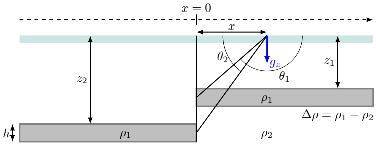
$\Delta\rho$ is the density contrast between the faulted unit and surrounding rock

G is the gravitational constant.

This expression can also be written as:

$$g_z = 2G\Delta\rho h\left[\theta_1 + \theta_2\right]$$

As before, the faulted horizontal unit is assumed to extend infinitely in the y direction (in and out of the $x - z$ plane). The faulted horizontal unit also extends infinitely in the x direction, away from the fault.



Prove to yourself that:

$$\theta_1 = \frac{\pi}{2} + \tan^{-1}\frac{x}{z_1}$$
$$\theta_2 = \frac{\pi}{2} - \tan^{-1}\frac{x}{z_2}$$

Gravity anomaly due to an offset unit across a dipping fault

The gravity anomaly due to an offset horizontal unit across a dipping fault is:

$$g_z = 2G\Delta\rho h\left(\pi + \tan^{-1}\left[\frac{x}{z_1} + \cot(\alpha)\right] - tan^{-1}\left[\frac{x}{z_2} + \cot(\alpha)\right]\right)$$

where:

α is the dip of the fault

z_1 and z_2 are depths to the top of the faulted horizontal unit in the hanging wall and footwall of the fault, respectively.

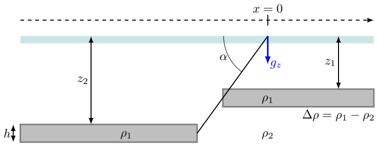
x is the horizontal offset from the intersection of the fault plane with the surface

h is the thickness of the horizontal unit

$\Delta\rho$ is the density contrast between the faulted unit and surrounding rock

G is the gravitational constant.

Note that the dipping fault, normal in this case, creates lateral offset of the faulted unit. The faulted edges of the unit remain square, rather than reflecting the dip of the fault.



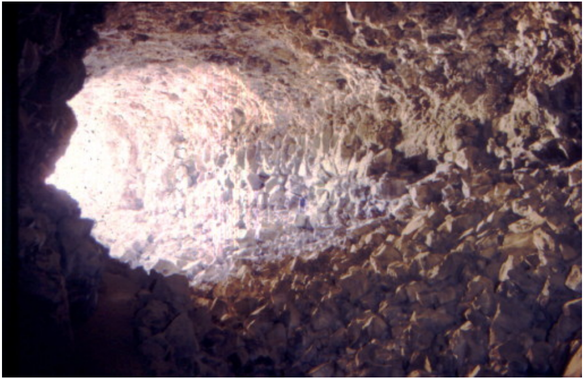
As before, the faulted horizontal unit is assumed to extend infinitely in the y direction (in and out of the $x - z$ plane). The faulted horizontal unit also extends infinitely in the x direction, away from the fault.

EOMA

1. In her paper on the gravity anomaly at Medicine Lake volcano, Carol Finn estimates that excess mass associated with a positive gravity anomaly at the volcano to be approximately 2×10^{17} g. Make this excess mass calculation yourself and compare your result to the one found by Finn and colleagues. Solve for the excess mass in the following steps:
   - Select the area of the anomaly and plot this map using GMT and the script provided (xcess_mass.pl). This might be very close to the area of the zoomed in map you prepared as part of Module 2. You will need to change values for these variables to zoom in: $west, $east, $south, $north, $tick_int (tick mark interval), $map_scale, $min and $max (min and max gravity values colored on the map), $cint (contour interval of map colors), $scale_anot_int (to adjust the label numbers on the color bar). Note that the map plots in UTM corrdinates, so the values given are in meters.
   - Once you have the anomaly map plotted you need to select a background gravity value for the plot. The excess mass will be associated with departure from this value. Note that the background value does not need to be constant. For example, Finn and colleagues used a regional trend as the background value. In this exercise, using xcess_mass.pl, the background gravity value must be constant. Adjust the variable $gravity_threshold in xcess_mass.pl and re-run the code to get the excess mass.
   - Review the PERL code and compare to the module excess mass example. Describe the steps taken in the code to get from the input data file to the excess mass estimate.
   - Write a description of your calculation and make your comparison with the Finn results. What do you think accounts for differences in your estimate?

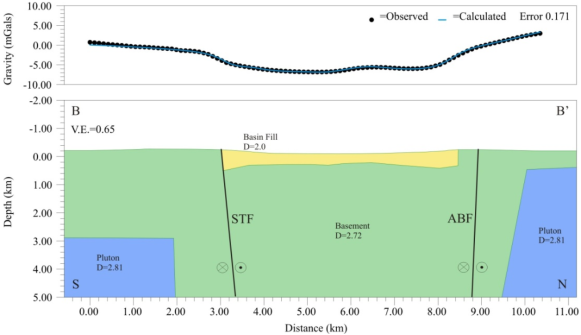
EOMA

2. Skull cave, lava beds national monument, is part of a lava tube system that extends along the lava flow for about 11 km. Where this picture was taken, the cave floor is about 35 m deep and the cave diameter is about 28 m. What is the expected gravity anomaly across Skull cave? Use a simple shape estimate to calculate the change in gravity across the cave. Discuss your assumptions and your answer. If you planned to collect a gravity profile across Skull cave, how would you plan to space your gravity readings?



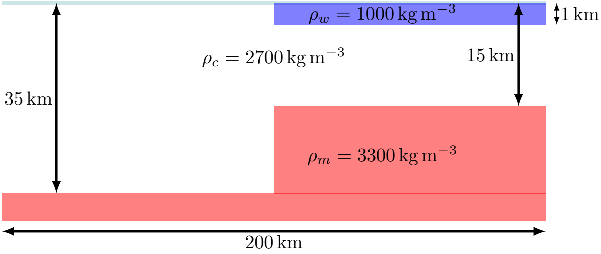
EOMA

3. Adam Springer (2008, Constraining basin geometry and fault kinematics on the Santo Tomás segment of the Agua Blanca Fault through a combined geophysical and structural study, Unpublished M.S. Thesis, University of South Florida, http://scholarcommons.usf.edu/etd/1779/), collected gravity data across a basin near Ensenada, Mexico that was formed as a pull-apart along a prominent strike-slip fault system. His gravity model is shown in the following figure. Note the density contrast between the basin-filling sediments and the surrounding basement rock is approximately $-720\,\mathrm{kg\,m^{-3}}$. Ignoring the plutons in Springer's model, estimate the gravity anomaly due to the basin-filling sediments using a simple shape. Draw your expected profile of gravity variation across the basin. Assume the basin length is much more than its width (the basin is "infinite" perpendicular to the profile. Discuss your answer. What re your assumptions? What is the effect of features not accounted for in your simple model?



EOMA

4. Gravity anomalies occur at continental margins. At passive margins, where the continental crust transitions to ocean crust without a plate boundary, gravity anomalies are caused by thinning of the crust, change in the density of the crust from continent to ocean, and the presence of the ocean itself. The following figure accounts for change in crustal thickness and the presence of the ocean in a simplified way (not drawn to scale). Calculate the gravity profile from continent to ocean, assuming this model is correct (Hint: make the depth to the "top" of the ocean slightly greater than 0, eg., $z = 1$ m). Discuss how you made this calculation and your assumptions. Compare with the gravity map of Florida in Module 2.



EOMA

5. Lots of geologic features can be modeled using simple shapes, as illustrated in questions 2–4. Find a geologic feature (that exists!) and model its expected gravity anomaly using a simple shape or combination of simple shapes. Possible features vary from planetary in scale to extremely small-scale features. Use the simple shape formulae given in the Module or find other examples of simple shapes in the literature.

Please describe the geologic feature, sketch the simple shape model you develop to estimate its gravity anomaly, calculate the anomaly - showing the script you use - and discuss your results and assumptions.