Magnetics

Introduction to Filtering using the Fourier Transform

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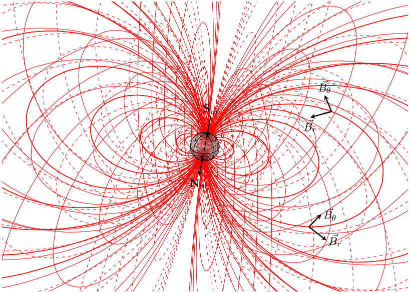
Potential Fields

Objectives for this week



Schematic Earth dipolar magnetic field. The field lines placed in the page plane are drawn as thick lines, those back with dashed lines and the field lines in front of the page with thin lines.

- Learn about the Fourier transform
- Learn about filtering profile data
- Use scripts to filter data using the Fourier transform

Fourier series

Any continuous (differentiable) function can be represented as an infinite series of sine and cosine terms. This idea was first developed by Joesph Fourier in the 18^{th}C while he worked out notions of the physics of sound. The basic idea is that any function that varies smoothly can be thought of as a *Fourier* series of sine and cosine terms with varying amplitudes and wavelengths:

$$
\begin{aligned}
f(x) = a_0 \quad &+ \quad \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi m x}{X}\right) \\
&+ \quad \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{X}\right)
\end{aligned}
$$

where a_m and b_n are the coefficients that need to be estimated to fit the function $f(x)$, m and n are termed wavenumber for the cosine and sine terms respectively, and X is the fundamental wavelength – the longest wavelength that might be considered for a specific problem.

The amplitude of each of the cosine and sine terms is controlled by the coefficients a_m and b_n. The wavelength represented by the cosine and sine terms varies as a function of m and n respectively.

Consider the terms $\frac{2\pi m x}{X}$ and $\frac{2\pi n x}{X}$. The wavenumber (m/X and n/X) is the number of wavelengths, or "cycles" per 2π units of x distance. X is the maximum wavelength (or maximum complete cycle) that can be represented on a given map or profile.

Example

Suppose a profile of magnetic observations is 100 m in length. For this profile, the fundamental wavelength is 100 m. $m = 1$ represents this longest wavelength for the cosine terms in the Fourier series. For each one unit step in x, $2\pi/100$ part of the cycle is completed (each step represents approximately 0.0628 radians per meter). For $m = 10$, one step in x represents approximately 0.628 radians per meter, a much shorter wavelength.

Wavelength and potential fields

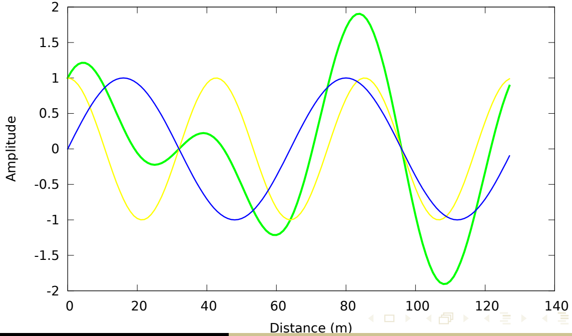
A Fourier series is a great way to represent potential field anomalies. Deep gravity or magnetic sources create broad anomalies, characterized by relatively long wavelengths and large coefficients (a_m and b_n) at small wavenumbers. Shallow sources of potential field anomalies are "short wavelength" and are characterized by large coefficients (a_m and b_n) at large wavenumbers.

Example

Consider the function:

$$f(x) = \sin\left(\frac{4x\pi}{N}\right) + \cos\left(\frac{6x\pi}{N}\right)$$

plotted as the thick green curve. N is the number of equally spaced sampling points that create the curve. In this case 128 samples are spaced at 1 m intervals. The function is the sum of two curves with different wavelengths (or wavenumbers). One curve, the sine function shown in blue, has a wavelength of 64 m. The other curve, the cosine function shown in yellow, has a wavelength of 128/3 m, or approximately 42.6 m.



Fourier transform

The power of Fourier series lies in the fact that it is possible to calculate the amplitude coefficients for the sine and cosine terms directly from the function, $f(x)$. The *Fourier transform* does this calculation. For a continuous function:

$$F(k) = \int_{-\infty}^{\infty} f(x) \exp(-i2\pi kx) dx$$

where k is the wavenumber, $f(x)$ is the function (e.g., change in magnetic field as a function of distance, x, and i is the imaginary unit, $\sqrt{-1}$.

$f(x)$ is a function in space, $F(k)$ is the same function in the wavenumber domain, x is a coordinate in space, k is the wavenumber:

$$k = \frac{1}{\lambda}$$

where λ is the wavelength. In the same way, any function $F(k)$ in the wavenumber domain can be transformed into the spatial domain.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(i2\pi kx) dk$$

Where did the sine and cosine terms go?

The notation used in this example takes advantage of trigonometric identities:

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\ \cos x + i \sin x &= e^{ix} \end{aligned}$$

where i is the imaginary unit. The Fourier transform of a real function, $f(x)$, is a complex conjugate, with real coefficients (the amplitudes of the cosine terms) and imaginary coefficients (the amplitudes of the sine terms).

Note that for $-\infty < x < \infty$ there is no fundamental wavelength. These equations apply to mathematical functions, but are less useful for geophysical series, like a magnetic profile, where the series is not infinitely long (the profile begins and ends) and samples are collected at discrete intervals.

The discrete 1D Fourier Transform for geophysical profiles

Fourier transforms of data sets, like magnetic profiles, are calculated using the discrete Fourier transform, which accounts for the fact that the data do not extend to infinity and that samples are collected at discrete intervals – so coefficients cannot be calculated for infinitely large wavenumbers. Formulas for the coefficients for finite data sets are:

$$\mathit{Re}(k) = \sum_{x=0}^{N-1} f(x) \cos\left(\frac{2\pi x k}{N}\right) \quad 0 \leq k \leq \frac{N}{2}$$

$$\mathit{Im}(k) = \sum_{x=0}^{N-1} f(x) \sin\left(\frac{2\pi x k}{N}\right) \quad 0 < k < \frac{N}{2}$$

or using the shorthand notation provided by trig identities:

$$F(k) = \sum_{x=0}^{N-1} f(x) \exp\left(\frac{2\pi x k}{N}\right) \quad 0 \leq \frac{N}{2}$$

$\mathit{Re}(k)$ and $\mathit{Im}(k)$ refer to the real (cosine) and imaginary (sine) coefficients, respectively, for a data set of N evenly spaced samples. As before, k/N is the wavenumber. $f(x)$ is the measured values as a function of distance along the profile.

Note that the real and imaginary components are only found to wavenumbers less than $N/2$. Shorter wavelength features, corresponding to higher wavenumbers, cannot be identified given the sample spacing. If shorter wavelength anomalies exist, they will appear to be longer wavelength anomalies – a condition called aliasing. Another way to think about this, from the perspective of a sampling strategy, is that it is impossible to identify an anomaly of wavelength less than twice the sample spacing. The Nyquist frequency is $1/2$ the sampling rate and is used to estimate the minimum wavelength anomaly detectable.

Example

Suppose a 100 m-long magnetic profile consists of 11 evenly spaced samples. What is the shortest wavelength anomaly that can be identified along the profile?

$$\frac{1\mathit{sample}}{10m}\frac{1}{2} = \frac{1}{20m}$$

or a minimum wavelength of 20 m. Magnetic sources that are shallow enough to produce anomalies with wavelength < 20 m will not be detected.

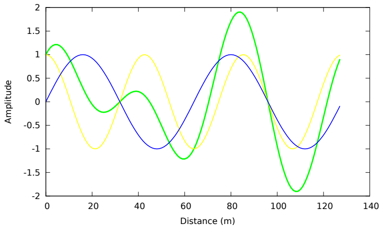
Power Spectrum

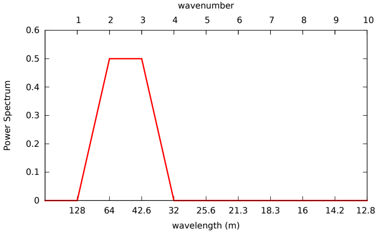
The real and imaginary coefficients of the Fourier transform are usually summarized using the amplitude spectrum:

$$|F(k)| = \sqrt{(ReF(k))^2 + (iImF(k))^2}$$

or the power spectrum:

$$|F(k)|^2 = (ReF(k))^2 + (iImF(k))^2$$



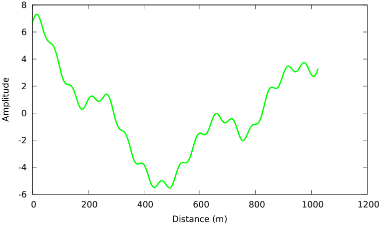


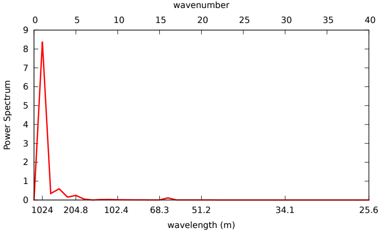
the Power Spectrum

The Fourier transform of geophysical data is usually shown as a power spectrum. Be sure you understand the relationship between the wavelength, wavenumber shown on the power spectrum graph and the function represented by the green curve at right, shown in the spatial domain. Note the amplitudes of the two curves that comprise the green curve are equal, leading to two equal-valued wavenumbers on the power spectrum.

Power Spectrum

Consider an example that looks a bit more realistic from the perspective of potential field data. The series shown at the right was constructed from a Fourier series of 11 sine and cosine terms, not very many in the scheme of things. Yet, a relatively complicated curve emerges. The Fourier transform of this curve is shown at left using a power spectrum. Note that most of the power (the largest amplitude wavenumbers) are a long wavelengths – corresponding to small wavenumbers. Nevertheless, you can see the contribution of relatively high wavenumbers in both the power spectrum and in the series in the spatial domain.





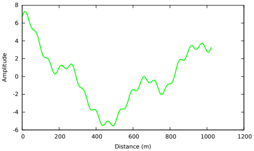
Anomaly wavelength and the power spectrum

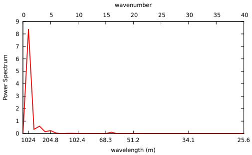
If the curve at left were actual gravity or magnetic data, it would be reasonable to conclude that there are different sources of the potential field anomaly, corresponding to different wavelengths (or wavenumbers) on the power spectrum. Relatively long wavelength variation must be associated with relatively deep sources. Short wavelength variation corresponds to relatively shallow sources. Thus, the power spectrum allows us to differentiate sources in magnetic or gravity data as a function of depth.

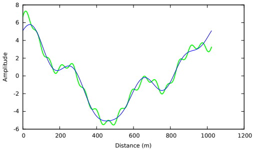
A first look at filtering

The true 'power' of the Fourier transform emerges when we consider how to use the power spectrum to filter potential field data. Consider the series (green curve) and its Fourier transform (red curve). How can we extract the relatively long wavelength anomalies associated with deeper sources from the relatively short wavelength anomalies, associated with shallow sources? The answer lies in the power spectrum. We can change the values of the *Re*(k) and *Im*(k) coefficients, shown together on the power spectrum, and transform the resulting series back into the spatial domain. This is an example of filtering.

The blue curve shows the filtered data set. To make the blue curve, *Re*(k) and *Im*(k) coefficients for $k > 6$ (wavelengths less than 1024/6 m or about 170 m), where all set to zero. This is an example of a low-pass filter (only relatively long wavelength features are transformed back in to the spatial domain).



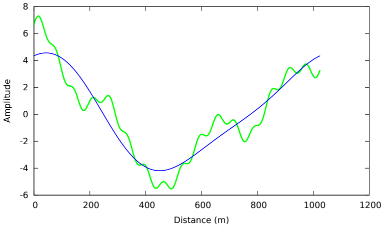




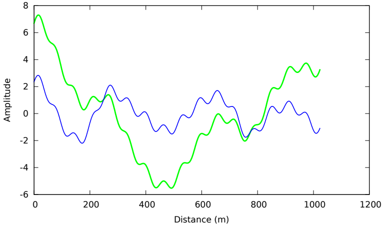
Inverse Fourier Transform

The Fourier transform is fundamentally like all mathematical transforms. The logarithmic transform allows you to take the logarithm, $y = log_{10}x$, of a number and transform it back again, $x = 10^y$. The Fourier transform is the same. The forward Fourier transform takes a series from space (or time) into the wavenumber (or frequency) domain. The inverse Fourier transform takes the series from the wavenumber domain to the spatial domain. So to perform the filtering operation, the forward Fourier transform was calculated, the data were filtered by changing real and imaginary coefficients of the series, and the inverse Fourier transform was calculated to put the resulting series back in the spatial domain.

Low-pass and high-pass filtering



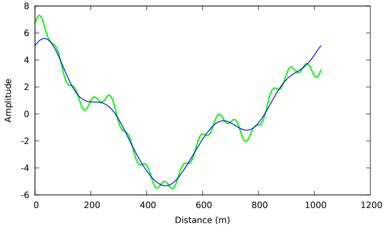
Here the same series is filtered using a low pass filter, cutting out all but the longest wavelength part of the signal. Only the fraction with wavelength greater than 341 m is passed through the inverse Fourier transform.



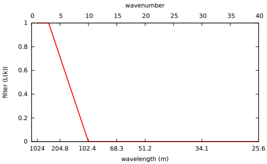
Here a high pass filter is used. Only the fraction of the series with wavelength less than 341 m is passed through the inverse Fourier transform.

Low-pass filtering

Myriad filter designs are possible. Here a low pass filter, $L(k)$, is used to attenuate (or eliminate) short wavelength variation in the signal. Note that a linearly ramped filter is used. This linearly ramped filter design changes the signal more gradually and can reduce spurious effects in the filtered data associated with abrupt changes in the power spectrum. At wavenumber values less than the long wavelength (low wavenumber) threshold, the signal is completely unaltered. At wavenumbers greater than the short wavelength (high wavenumber) threshold the signal is completely cut. The ramp is in-between these threshold values.



The original, unfiltered series is shown in green and the low-pass filtered series is shown in blue.



The filter design for this low pass filter is ramped, passing long wavelengths unaltered (1) and completely attenuating short wavelengths (0). At intermediate wavelengths, the signal is attenuated linearly as a function of wavelength. The wavenumber is shown as integers of $1/\lambda \times L$, where L is the number of samples in the series times the sample spacing.

Band-pass filters

A band-pass filter is one that cuts both short wavelength and long wavelength parts of the signal. Linearly ramped band-pass filters are characterized by four threshold values.

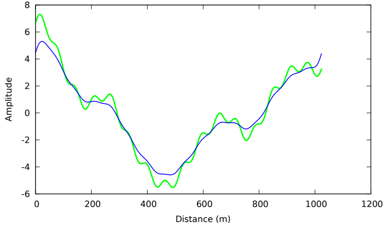
Upward continuation

Imagine that you collect a profile of magnetic data on the ground surface, then collect magnetic data along the same profile, but from an airplane at 1000 m elevation. The aeromagnetic survey is further from the source of magnetic anomalies, so the aeromagnetic profile will be characterized by lower amplitude and longer wavelength anomalies than the ground profile. Because there are no magnetic sources in the air, we can determine what the profile will look like at 1000 m elevation given the variation in the magnetic field observed along the ground profile. Mathematically, to calculate the profile at 1000 m height, given the ground profile, requires *convolution* of the ground profile with another function. This convolution operation is simple using the Fourier transform:

$$U(k) = F(k)e^{-zk}$$

where $F(k)$ is the Fourier transform of the ground magnetic profile data, $f(x)$; k is the wavenumber $(1/\lambda)$ where λ is the wavelength; z is the distance of the upward continuation (e.g., 1000 m). Notice that convolution is a multiplication in the Fourier domain.

The inverse transform of $U(k)$ is $u(x)$, the upwardly continued profile. For small values of k (corresponding to long wavelengths), e^{-zk} is close to one and there is relatively little difference between $U(k)$ and $F(K)$. As the wavenumber gets larger (shorter wavelengths), $e^{-zk} \to 0$.



Here the original data set ($f(x)$, green) is upwardly continued by 100 m ($u(x)$, blue). Notice that the relatively short wavelength features of the profile are strongly attenuated, and the long wavelength features of the profile are relatively unchanged. The amplitude of all wavelengths in the green profile are reduced by upward continuation, but the magnitude of the reduction is strongly dependent on wavelength.

Derivative

Recall this important characteristic of periodic functions:

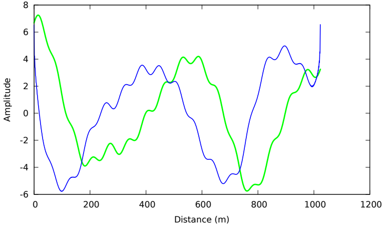
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\sin x = \cos x$$

This relation suggests a procedure to take the derivative of any function using the Fourier transform. For each wavenumber, k, the coefficient of the real term becomes the negative of the coefficient of the imaginary term, and the coefficient of the imaginary term becomes the real term. This 'switch' is represented algorithmically:

$$f(x) \Rightarrow F(k)$$
$$Re'(k) = -Im(k)$$
$$Im'(k) = Re(k)$$
$$F'(k) \Rightarrow \frac{d}{dx}f(x)$$

where the double arrow indicates the forward and inverse Fourier transforms.

The first derivative of a magnetic profile can help us visualize the location of potential anomalies, since inflection points on negative slopes (largest negative first derivative) roughly coincide with the center of the anomalous body at magnetic inclinations $0^\circ < I < 90^\circ$.



The second derivative can also be found with a Fourier transform, since:

$$\frac{d^2}{dx^2}\cos x = -\cos x$$
$$\frac{d^2}{dx^2}\sin x = -\sin x$$

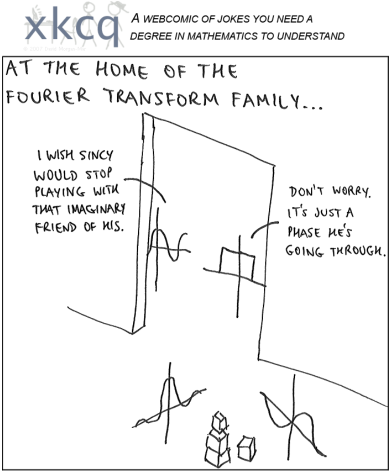
Some practical aspects of filtering with the Fourier transform

There are some important practical aspects to working with Fourier transforms and filtering data using these methods.

- Usually an algorithm is used to calculate the forward and inverse Fourier transforms. This algorithm is called the fast Fourier transform (FFT). In a practical sense, there is no difference between a Fourier transform and FFT, except that the former refers to a mathematical transformation and the latter refers to a computational method to accomplish this transform.
- Numerous computer languages and software packages are available to do FFTs and filtering operations. Unfortunately, these do not always use the same nomenclature, especially for indexing data. In potential field geophysics, we are almost always concerned with filtering data that are 'real' (have no imaginary component) and have been collected as some sampling interval. When the data are transformed, a real, discrete FFT algorithm is used (note that transformed data do have real and imaginary components!).
- One characteristic of FFTs is that they require 2^n equally-spaced samples along the profile (or in the series), where n is an integer (e.g., 128, 512, or 1024 samples). Often data must be interpolated to equal spacing, for example if there are missing values. The series can be 'padded' to reach 2^n samples by adding zeros (zero padding), or by repeating the series.
- Trends in data must be removed before the FFT. Trends in data are essentially associated with longer wavelength cycles than the length of the profile. Usually a linear trend is removed before the FFT.
- Many additional filtering operations exist than are discussed in this module. For magnetics these include reduction-to-the-pole (using a Fourier transform to make the data appear they were collected at the magnetic north pole, assuming an angle of inclination), and the pseudo-gravimetric transform (using the reduction to pole and additional constants to transform the magnetic field (nT) to a gravity field (mgal) assuming a simple relation between magnetic and density properties of rocks).

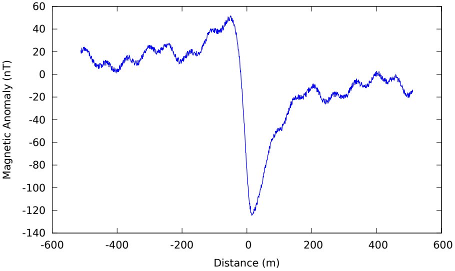
Answer the following 6 or 7 questions...

1. Suppose you collect 1024 observations of the magnetic field along a single, straight profile at one meter intervals along the ground. What are the maximum and minimum anomaly wavelengths you can observe in this profile?
2. You upwardly continue the magnetic profile. Draw a graph showing the coefficients of this convolution as a function of wavenumber. That is, graph $C = e^{-zk}$, for $k = (1/1024, 2/1024, 3/1024, ..., 1)$. Make this plot for different values of z and explain what the graph means in terms of the magnetic anomalies you might observe along the profile at different elevations.



EOMA

Consider a N-S-trending line of magnetic data:



Use the supplementary information to filter this series, complete the following tasks.

3. Plot the profile. The magnetic data are contained in the file *dike.dat*. Provide a figure caption that includes brief discussion of the anomaly(ies) you observe on this profile.
4. Calculate and plot the power spectrum of the magnetic profile using the PERL script *spectrum.pl*. Write a figure caption for this plot, explaining the power spectrum and what it means in terms of the observed magnetic anomlay(ies).
5. Write a code to upwardly continue the magnetic profile. Use the example in supplementary material to help you write the code. Plot the resulting profile along with the original profile on the same plot. Experiment with different amounts of upward continuation (z) and determine which values filter the data best, in your opinion (it is a matter of opinion!) and explain why in the figure caption.

EOMA

6. Write a perl script to low-pass-filter the magnetic profile. Use the example in supplemenatry material to help write this code. Plot the resulting profile along with the original profile on the same plot. Experiment with different low pass filters and determine which values filter the data best, in your opinion (it is a matter of opinion!) and explain why in the figure caption.
7. (extra credit) When filtering data, it is always a great idea to plot what you have filtered out (removed from the profile) in addition to what has passed through the filter. Plot the residuals (difference between the original and filtered data), for your upwardly continued and low-pass filtered profiles. Hey – one person's noise is another person's alleatoric masterpiece.